Last time: If $V_1\,dx + V_2\,dy + \cdots = d(g)$ ie, vector field $(V_1, V_2, \ldots)$ is conservative, then

$$\oint_C V_1\,dx + V_2\,dy = g(b) - g(a)$$

$b$: end point, $a$: beginning pt.

Consequence. If $C$ is a closed loop.


end point $= a =$ beginning pt.

If $V$ is conservative, then

$$\oint_C V \cdot ds = 0.$$

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Question: If we have a closed loop, what if the vector field is not conservative? Is there a slick way to compute the line integral?

# Interlude: Vector fields, Differential Forms, and derivatives of vector fields & differential forms.

Vector fields $\Longleftrightarrow$ 1-forms $\Longleftrightarrow$ 2-forms ($\mathbb{R}^3$)

components $V = (V_1, V_2, V_3)$

$dx \wedge (\text{that}) = dx \wedge dy \wedge dz$

$$V_1 \longrightarrow V_1\, dx \qquad V_1\; dy \wedge dz$$

$$V_2 \longrightarrow V_2\, dy \qquad (dy \wedge dz \wedge dx = -dy \wedge dx \wedge dz = dx \wedge dy \wedge dz) \qquad V_2\; dz \wedge dx$$

$$V_3 \longrightarrow V_3\, dz \qquad V_3\; dx \wedge dy$$

$$(V_1, V_2, V_3) \qquad V_1\,dx + V_2\,dy + V_3\,dz$$

$$\downarrow$$

$$V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy.$$

$$V_1\,dx + V_2\,dy + V_3\,dz \xrightarrow{\ d\ } d(V_1\,dx + V_2\,dy + V_3\,dz)$$

$$d(V_1\,dx + V_2\,dy + V_3\,dz)$$
$$= \left(\cancel{(V_1)_x\,dx} + (V_1)_y\,dy + \underline{(V_1)_z\,dz}\right) \wedge \underline{dx}$$
$$+ \left((V_2)_x\,dx + \cancel{(V_2)_y\,dy} + (V_2)_z\,dz\right) \wedge dy$$

$$+\left((V_3)_x\,dx + (V_3)_y\,dy + \cancel{(V_3)_z\,dz}\right)\wedge dz$$

$$\boxed{= \left((V_3)_y - (V_2)_z\right) dy\wedge dz + \left((V_1)_z - (V_3)_x\right) dz\wedge dx + \left((V_2)_x - (V_1)_y\right) dx\wedge dy}$$

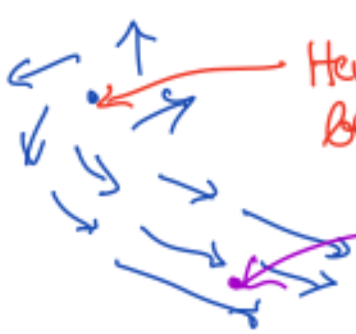
$$(V_1, V_2, V_3) \longmapsto \left((V_3)_y - (V_2)_z,\ (V_1)_z - (V_3)_x,\ (V_2)_x - (V_1)_y\right)$$

This is the same as

$$\nabla \times V = \underline{\text{curl}\, V}$$

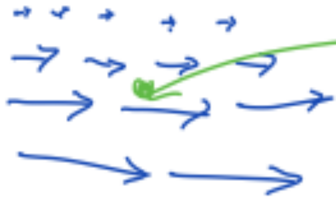
$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\partial_y V_3 - \partial_z V_2\right) - \hat{j}\left(\partial_x V_3 - \partial_z V_1\right) + \hat{k}\left(\partial_x V_2 - \partial_y V_1\right)$$

Measures how the vector field is twisting (Right hand rule) around each point



Here $\text{curl}(V) = \nabla \times V$ would be a vector pointing upward.

Here $\text{curl}\, V = 0$.



Here curl $V$ would also be a vector pointing up (an object would twist counter clockwise under this flow)

If $V$ is conservative, i.e. $V = \nabla g$, then $\nabla \times V = 0$ vector $= (0,0,0)$

(we have $\nabla \times (\nabla(g)) = 0$.

differential forms: $d(dg) = 0$.)

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Thinking of $V = (V_1, V_2, V_3)$ as a 2-form ($\tilde{V} = V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy$)

$$d\tilde{V} = d(V_1) \wedge dy \wedge dz + d(V_2) \wedge dz \wedge dx + d(V_3) \wedge dx \wedge dy$$

$$\left( (V_1)_x\, dx + \cancel{(V_1)_y\, dy} + \cancel{(V_1)_z\, dz} \right)$$

$$= (V_1)_x\, dx \wedge dy \wedge dz + (V_2)_y\, dy \wedge dz \wedge dx + (V_3)_z\, dz \wedge dx \wedge dy$$

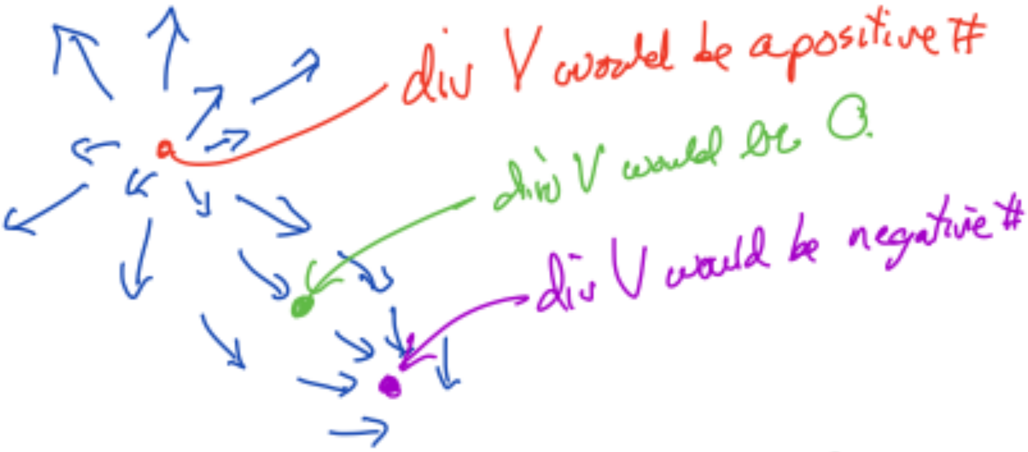
$$= \underbrace{\left( (V_1)_x + (V_2)_y + (V_3)_z \right)}_{\text{function}} dx \wedge dy \wedge dz$$

$$\operatorname{div} V = (V_1)_x + (V_2)_y + (V_3)_z \quad \text{function}$$

<u>divergence</u>

$$= \nabla \cdot V = (\partial_x, \partial_y, \partial_z) \cdot (V_1, V_2, V_3)$$

This number at each point measures how much the vector field is expanding




$$\text{Since } d(\underbrace{\underbrace{d\overset{\text{1-form}}{\omega}}_{\text{curl}}}_{\text{div.}}) = 0$$

We always have $\operatorname{div}(\operatorname{curl} W) = 0$ for any vector field $W$.

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<u>Examples</u> Let $V = (x^2+y, x-z, z+2x)$

$$\operatorname{div} V = \nabla \cdot V = (x^2+y)_x + (x-z)_y + (z+2x)_z$$

$$= 2x + 0 + 1 = \boxed{2x+1}.$$

$$\operatorname{curl} V = \nabla \times V = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ (x^2+y) & (x-z) & (z+2x) \end{vmatrix}$$

$$= \hat{i}\big(\partial_y(z+2x) - \partial_z(x-z)\big)$$
$$- \hat{j}\big(\partial_x(z+2x) - \partial_z(x^2+y)\big)$$
$$+ \hat{k}\big(\partial_x(x-z) - \partial_y(x^2+y)\big)$$

$$= \hat{i}\big(0 - (-1)\big) - \hat{j}(2-0) + \hat{k}(1-1)$$

$$= \hat{i} - 2\hat{j} + 0 \Rightarrow (1,-2,0)$$

usually you get functions!

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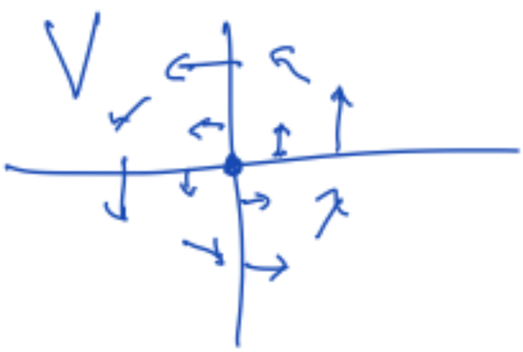
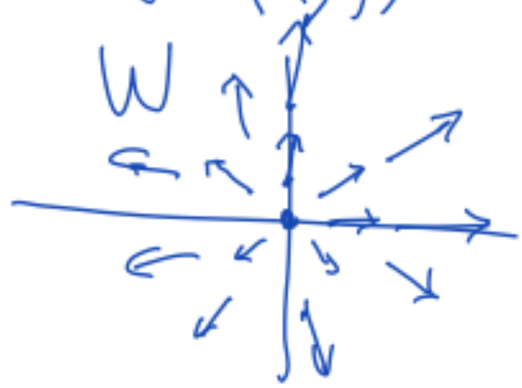
**Example** $V = (-y, x)$ $\quad W = (x, y)$




$$\operatorname{div} V = \partial_x(-y) + \partial_y(x) = 0$$

$$\operatorname{curl} V = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \hat{i}\big(\partial_y(0) - \partial_z(x)\big) - \hat{j}\big(\partial_x(0) - \partial_z(-y)\big)$$
$$+ \hat{k}\big(\partial_x(x) - \partial_y(-y)\big) = 2\hat{k} = (0,0,2)$$

$$\operatorname{div} W = \partial_x(x) + \partial_y(y) = \boxed{2}$$

$$\operatorname{curl} W = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & 0 \end{vmatrix} = \hat{\imath}\left(\partial_y(0) - \partial_z(y)\right) - \hat{\jmath}\left(\partial_x(0) - \partial_z(x)\right) + \hat{k}\left(\partial_x(y) - \partial_y(x)\right)$$

$$= \boxed{(0,0,0)}.$$

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Back to $\displaystyle\int_C V \cdot ds = \int_C V_1\,dx + V_2\,dy + V_3\,dz$

If $C$ is a closed loop surrounding a region $R$ in CCW direction


$C = \partial R$

("boundary of $R$")

$$\int_C V \cdot ds = \int_C V_1\,dx + V_2\,dy + V_3\,dz = \int_{\partial R} V_1\,dx + V_2\,dy + V_3\,dz$$

$$= \iint_R d(V_1\,dx + V_2\,dy + V_3\,dz)$$

Stokes' Thm
(Green's Thm – 2D)

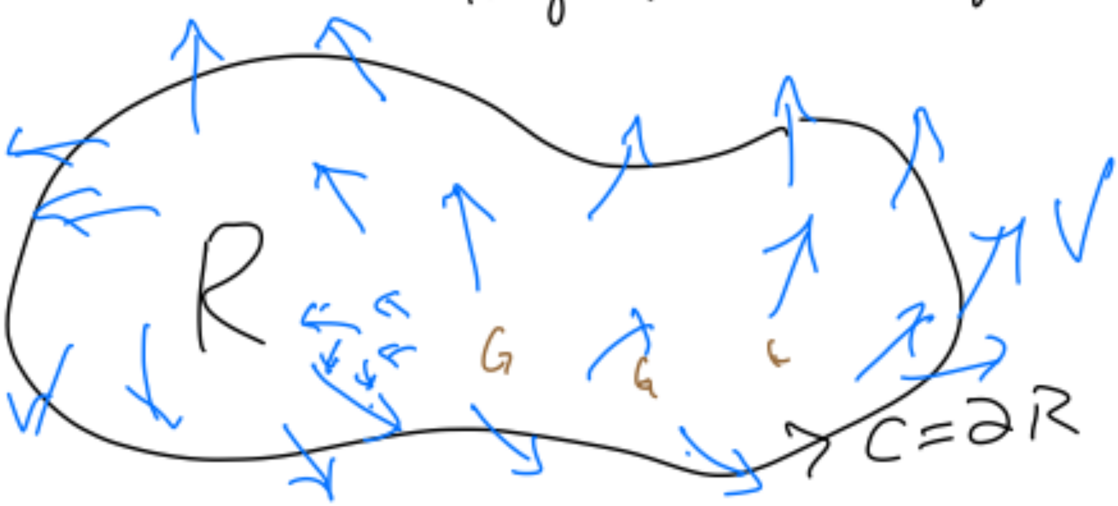
convert to vector field language

$$= \iint_R (\text{curl } V) \cdot N \, dA$$

$N_1\,dy\wedge dz + N_2\,dz\wedge dx + N_3\,dx\wedge dy$

unit normal to R

"flux of the curl through R"




$\int_C V \cdot ds$ ← how much the vector field V is pushing along C

$= \iint_R (\text{curl } V) \cdot N \, dA$ ← adds up the ccw twisting of V on the inside R.

$$V_1\,dx + V_2\,dy + V_3\,dz = \alpha$$

$$\int_{\partial R} \alpha = \int_R d\alpha$$